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Ebunoluwa Adebiyi[1]
Dr Ini Adinya[2]
Paul Kehinde I.[3]

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[1] University of Ibadan
[2] University of Ibadan
[3] University of Ibadan

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A Fractional-Order Grey Model with Fiscal Drag for Tax Revenue Forecasting in the Nigerian Public Sector

Ebunoluwa Adebiyi *[1], Dr Ini Adinya †[2], and Paul Kehinde I. ‡[3]

[1,2,3]University of Ibadan

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Abstract

This article explores the potential of the Fractional Grey Model (FFGM) to forecast the fiscal impact of skill enhancement programs for Nigeria's civil servants in CONPSS levels 07, 08, and 10. By upskilling these workers, who represent the largest population in the civil service, the government can increase promotions, salaries, and consequently, tax revenue. The article narrates how investing in human capital can create a positive fiscal feedback loop, benefiting both workers and the state.

1 Introduction

In many developing economies, including Nigeria, public sector workforce planning and financial forecasting are often hindered by the scarcity of reliable data, institutional opacity, and systemic uncertainty. One critical yet understudied segment in this context is the mid-level civil service workforce, particularly personnel within the Consolidated Public Service Salary Structure (CONPSS) Levels 07–10. These workers, encompassing administrative officers, clerks, and registry staff, constitute a substantial portion of the civil service. Despite their numerical strength and potential for economic impact, their career progression and income growth have remained largely stagnant due to rigid promotion structures and limited access to professional development opportunities. Unlike their counterparts at Level 15, who often secure their roles through political appointments and are relatively few in number, these mid-level workers are governed by a rigid, seniority-based promotion system that offers little room for upward mobility. This structural immobility requires a unique form of "fiscal drag" through underutilized talent and access to professional development to enhance both tax generation and national productivity, rather than focusing disproportionately on the few appointees at higher levels. This paper introduces a broader concept of how the government should strategically invest in and leverage this large population of civil servants at Levels 07–10 to enhance both tax generation and national productivity, rather than focusing disproportionately on the few appointees at higher levels. Addressing these challenges requires accurate forecasting tools capable of functioning effectively under the very conditions that characterize public sector systems: incomplete information, small sample sizes, and data irregularities. Traditional statistical models often fall short in such contexts due to their dependency on large, high-quality datasets. To bridge this gap, Grey System Theory, introduced by Deng Julong in the early 1980s, offers a robust framework for modeling and forecasting systems plagued by uncertainty, incomplete information, and limited data. The Grey Model GM(1,1) is one of the most widely used variants due to its simplicity and efficacy in forecasting time series, especially in scenarios with small sample sizes or poor data quality. However, the traditional grey model relies on integer-order accumulation and differentiation, which may limit its ability to capture complex dynamics such as long-term dependencies and memory effects commonly observed in economic and financial data. (JUSTIFICATION OF CONPSS SCALES INLINE WITH BANK) To overcome these limitations, researchers have integrated concepts from fractional calculus into grey modeling, leading to the development of Fractional Grey Models (FGMs). Fractional calculus extends differentiation and integration to non-integer (fractional) orders, allowing for more flexible and accurate modeling of dynamic processes. In FGMs, fractional-order derivatives and fractional accumulated generating operations (FAGO) are used to more effectively characterize the underlying patterns and stochastic behaviors in data, enhancing the predictive performance over classical models.

This enhancement is particularly significant in economic applications such as tax revenue prediction, where data are often noisy, incomplete, and influenced by numerous interdependent factors. Tax revenue streams can exhibit complex temporal dynamics driven by policy changes, economic cycles, and taxpayer behavior, making traditional forecasting methods less reliable. The memory effect introduced by fractional derivatives enables FGMs to capture these long-term influences, providing a more nuanced and realistic modeling approach.

By applying fractional grey models to tax revenue forecasting, governments and financial institutions can gain im-

*gift.adebiyi5@gmail.com
†Iniadinya@gmail.com
‡pimoukhedeme@helpmaninstitute.org

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proved prediction accuracy with relatively small datasets. This can assist in budgeting, fiscal planning, and policy evaluation under uncertainty, ultimately contributing to more efficient resource allocation and economic management.

The fractional-order grey model represents a significant advancement in time series forecasting, merging grey system theory with fractional calculus to better model complex and uncertain systems, such as tax revenue. This synergy enables more accurate, flexible, and reliable predictions critical for informed decision-making in public finance and economic policy.
Fractional accumulation represents a significant leap in the sophistication of forecasting models, particularly in contexts where data is limited and prone to noise—an issue frequently encountered in energy consumption analysis. Traditional models often smooth data in discrete, all-or-nothing steps—what you might call "integer order" operations—essentially forcing the data to conform to rigid, pre-set transformations. This approach, while sometimes effective, falls short when faced with the subtle dynamics and nonlinear behavior typical of, say, regional energy usage or small-sample economic systems.

Duan et al. [1] make a notable advance in this domain by re-engineering grey system theory to incorporate fractional calculus and evolutionary optimization techniques. By introducing a fractional-order accumulation operator, defined via the gamma function, their methodology transcends the conventional model's stepwise data transformation. Instead, it offers a granular continuum of smoothing and memory effects, precisely tuning the model's response to idiosyncrasies within the dataset. To put it another way, this framework enables the model to capture an entire spectrum between raw, unprocessed data and the heavily-aggregated transformations produced by traditional integer-order accumulations.

The practical upshot is best demonstrated in Duan et al.'s application of the FSIGM (Fractional-order Self-adaptive Intelligent Grey Model) to China's crude oil consumption data. The classic SIGM model, constrained by statistical requirements such as a class ratio test, often excludes non-conforming datasets—an all-too-common occurrence in the volatile arena of energy consumption. Through fractional-order extension, Duan et al. circumvent this restriction entirely, allowing the FSIGM to flexibly adapt to exactly the kind of small, noisy samples that stymie less sophisticated approaches. In empirical tests spanning 2002 to 2014, the FSIGM consistently yielded lower mean absolute percentage error (MAPE) values than established models such as GM(1,1), DGM, NDGM, and the earlier version of SIGM, both within the training set and in out-of-sample forecasts.

Meanwhile, Liang Zeng and colleagues introduce yet another layer of methodological innovation: the integration of time delay characteristics with fractional-order accumulation. The classic GM(1,1) model assumes that all historical data points contribute equally to current outcomes. Yet, in complex systems like regional energy consumption, lagged effects—where past consumption influences future trends—are the rule rather than the exception. Zeng's model, dubbed $NGM\mu(1,1,\tau,r)$, accounts for this by embedding a time delay parameter and optimizing fractional-order and power term parameters using particle swarm optimization. This permits the model to dynamically "weigh" different periods of historical data, capturing both immediate and lagged impacts on consumption patterns.

Crucially, the time delay mechanism is not added arbitrarily. Grey absolute correlation analysis is employed to determine the statistically optimal delay (τ), while particle swarm optimization—a population-based, stochastic algorithm—simultaneously fine-tunes the other hyperparameters. In their evaluation using Guangdong's primary energy consumption data (2006–2017), this model achieved MAPE scores as low as 0.39% for fitting and 1.83% for forward prediction, outperforming all tested alternatives by a sizeable margin. These figures underscore the ability of the proposed framework to manage the uncertainty and irregularity that characterize real-world energy datasets.

The broader trend reflected by both Duan et al. and Zeng's [?] studies is unmistakable: the fusion of grey system theory with fractional calculus, memory- and delay-based mechanisms, and modern optimization algorithms is rapidly elevating the accuracy and applicability of prediction models within fields where small or imprecise datasets are the norm rather than the exception. This is not a trivial development; traditional data-driven models often founder in such scenarios due to insufficient sample sizes or undue sensitivity to noise. The current wave of research essentially redefines what is possible in predictive analytics for uncertain and nonlinear systems.

It is also worth noting the direction of future investigation highlighted by these researchers. For instance, Zeng's team identifies the refinement of time delay parameter determination as a critical avenue for further improvement, acknowledging that even greater precision and adaptability may yet be achievable. Moreover, the expansion of these models toward hybrid frameworks—incorporating additional sources of auxiliary information or alternative optimization routines—suggests an ongoing evolution marked by ever-increasing complexity and power.

Finally, Yang's recent efforts to inject advanced meta-heuristic optimization methods—such as a fractional-order, Gaussian-mutation-augmented variant of the jellyfish search algorithm—demonstrate the continued cross-pollination of ideas across mathematical, computational, and engineering disciplines. The result is a modeling landscape that is both theoretically robust and eminently practical for real-world problems, ranging from rural income prediction to large-scale primary energy forecasting. In summation, the confluence of fractional calculus, grey systems theory, time-delay mechanisms, and evolutionary optimization constitutes a vibrant and rapidly progressing frontier in statistical modeling, particularly for small, uncertain, and complex datasets.

Guo (2021) offers a detailed examination of China's impending population transformation, undertaking the ambitious task of forecasting demographic trends between 2015 and 2050. The author employs grey fractional-order models—a mathematically nuanced branch of grey system theory designed for situations loaded with incomplete or ambiguous data. In particular, the study leverages the fractional-order GM(1,1) model for short-range projections from 2020 to 2025 and the more sophisticated fractional-order Verhulst model for medium- and long-term horizons, stretching out to 2050. This bifurcated approach acknowledges that population change operates at multiple temporal scales and is

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influenced by a tangled web of economic, social, and policy-driven variables.

The projections themselves are sobering and nuanced. Anticipating an inverse S-shaped growth trajectory, Guo predicts that China's population will peak at roughly 1.433 billion by 2050. Underneath this headline finding are a number of demographic currents: birth rates and the natural population growth rate are likely to decline, while death rates may remain comparatively stable. Perhaps the most pressing theme, however, is the rapid aging of the population. The forecast points to a marked increase in both the sheer size of the elderly cohort and the dependency ratio, implying significant financial and social demands on working-age citizens and state resources. Notably, Guo's analysis highlights the technical capability of fractional-order accumulation to suppress randomness in the dataset—a crucial step, given the complex and noisy nature of demographic data. Particle swarm optimization (PSO) is employed for parameter calibration, and multiple error metrics confirm that the models achieve high reliability. The study concludes by underlining the urgency of robust policy responses, particularly around education, labor market adaptation for older workers, and proactive strategies to confront demographic aging and its socio-economic repercussions.

Moving to Tu, Chen, and Wu (2021), their contribution is notable for its methodological innovation—they introduce the "aging fractional accumulation operator," an upgrade to traditional grey forecasting paradigms. By allowing dynamic adjustment of data accumulation weights, their AGM(1,1) model addresses one of the central limitations of classic grey models: inflexibility in the face of irregular, limited, or noisy data. Their model's adaptive weighting is particularly valuable in scenarios where historical trends are erratic or when only a small sample is available, contexts often encountered in real-world socio-economic forecasting. Through a series of case studies, the researchers demonstrate that their enhanced model consistently surpasses traditional methods in predictive accuracy, setting a potential new standard for applied grey system modeling.

Yu and Xiang (2019) extend the scope of grey modeling to fiscal data, specifically enhancing the fractional-order FAGM(1,1) model with the integration of the Cotes integral formula—an advancement that improves both the theoretical underpinnings and practical application of the model. Utilizing government fiscal expenditure data, their work rigorously compares the novel CFAGM(1,1) model against established benchmarks, finding clear, measurable improvements in predictive power. The significance of this work lies not only in technical validation but also in its capacity to inform real-world activities such as macroeconomic planning, budget allocation, and financial regulation.

Lastly, the study by Chen, Gong, Li, and Guo (2024) signifies a leap in the modeling of complex resource systems with their introduction of the IBCFGMP (1,1,N) grey model. This model's sophistication comes from its multi-pronged optimization, attacking the challenge from several angles: background value selection, initial condition calibration, fractional-order fine-tuning, and optimization of grey action quantities. By using advanced algorithms such as particle swarm optimization, the model attains greater universality and stability, demonstrably outperforming traditional grey models when applied to China's energy consumption forecasts. Their precise predictions for hydroelectric, nuclear, and coal energy consumption up to 2026 are poised to offer invaluable guidance for both policymakers and industry actors tasked with resource management and sustainability planning. Importantly, the authors are transparent about current limitations, specifically the univariate nature of their model and its inability to seamlessly incorporate temporal lags evident in empirical data. This points to an ongoing evolution in the field—as models grow more complex, so too do their ambitions to mirror the multifaceted reality they seek to predict. In summary, these varied but related studies collectively push the boundaries of grey modeling in socio-economic and energy systems forecasting. They underscore the need for methodological adaptability, rigorous parameter tuning, and constant validation against empirical realities, providing a robust foundation for strategic decision-making in contemporary China.

Ruixiao Huang's 2022 study stands out in cybersecurity forecasting, tackling the persistent challenge of small, irregular data sets with an innovative methodological twist. Instead of relying on the classic grey model—GM(1,1), which, frankly, just slaps the same weight on every historical data point—Huang's team pushes forward with a fractional accumulative grey model (FAGM). The significant shift here lies in its use of fractional-order accumulation, which doesn't just passively collect history but actually elevates the importance of new data. This choice is not just mathematically clever; it actually mimics how real-world information often unfolds, where fresh trends deserve more attention than stale, outdated events.

But the innovation doesn't stop at the model structure. The process of optimizing the fractional order—a fiddly and crucial hyperparameter—often stumps researchers because it's not something you can just eyeball or pick out of a textbook. Huang et al. circumvent this hurdle by combining genetic algorithms with particle swarm optimization (GA-PSO), borrowing ideas from both evolutionary biology and swarm intelligence. This hybrid approach, which the researchers cleverly dubbed GAPSO-FAGM(1,1), methodically scours the search space for the best fractional order, ensuring that the model is neither stuck in a rut nor off chasing random noise.

Empirical validation is robust: applying GAPSO-FAGM(1,1) to weekly cybersecurity incident reports sourced from China's CNCERT/CC, the model doesn't merely edge past its predecessors; it demonstrates clear superiority in terms of both predictive accuracy and consistency. Compared to staples like GM(1,1), its multi-variable cousin GM(1,n), and even the fractional discrete grey seasonal model FDGSM(1,1), the proposed hybrid model maintains its performance edge, particularly under the stress of limited or noisy data conditions—precisely the type of environment where deep learning models tend to flounder due to their data-hungry nature.

What stands out in Huang's analysis is not just the technical ingenuity, but an acute sensitivity to the practical realities of cybersecurity monitoring. In practical terms, this means smaller organizations—often lacking in vast troves of labeled data—can now potentially leverage more dependable forecasting methods. The research team isn't oblivious to their own limitations: while their model is a notable step forward, the quest for even greater prediction stability continues. Additionally, there's a nod toward the future, with ambitions to weave middle- and long-term forecasting capabilities into the approach, perhaps by marrying this grey modeling lineage with new deep learning techniques as datasets grow richer.

In sum, Huang's study does more than introduce another algorithm: it meaningfully advances the state of predictive analytics for cybersecurity, demonstrating that with the right tools—fractional modeling, smart optimization, and a nuanced appreciation for real-world constraints—forecasting in uncertain environments is not just possible but can be made reliably actionable. However, the classical GM(1,1) model operates on the basis of integer-order accumulation and differentiation, which may not adequately capture the long-range dependencies, nonlinearities, and memory effects often embedded in complex economic behaviors such as salary progression and bureaucratic promotion systems. These limitations reduce the model's sensitivity to subtle structural dynamics and can compromise forecasting accuracy in scenarios involving prolonged institutional inertia or gradual systemic change.

To overcome these constraints, this study proposes the development of a Fractional Grey Model (FGM) with a Fiscal drag that will enhance variant of the traditional GM(1,1) that incorporates fractional-order accumulation and differential operators. By introducing fractional calculus into the grey modeling framework, the FFGM provides greater flexibility in simulating dynamic memory, smoothing irregularities, and modeling long-term dependencies inherent in economic time series. This refinement is expected to yield more accurate and adaptive forecasts, particularly for projecting career and income trajectories of civil servants within CONPSS Levels 07–10. In doing so, the model can support more informed policy decisions related to human capital planning, taxation potential, and public sector reforms.

2 Model Framework

The Fractional Grey Model (FFGM(1,1,i)) with fiscal drag extends the classical GM(1,1) model by incorporating fractional-order accumulation. This provides greater flexibility and improved accuracy for modeling non-homogeneous exponential sequences.

we have assumed that $X^{(1)}$ is 1-AGO. The r-order cumulative generation sequence is defined below.

Definition 2.1 (Fractional Order Accumulation)**.** *Let* $X^{(r)} = \left\{x^{(r)}(1), x^{(r)}(2), \dots, x^{(r)}(n)\right\}$ *be the r-AGO sequence, where:*

$x^{(r)}(k) = \sum_{i=1}^{k} x^{(r-1)}(i) = \sum_{i=1}^{k}\sum_{j=1}^{i} x^{(r-2)}(j), \quad r \in \mathbb{R}^+,\ k = 1, 2, \dots, n$

can be equivalently expressed as:

$x^{(r)}(k) = \sum_{i=1}^{k} \frac{(k-i+1)(k-i+2)\cdots(k-i+r-1)}{(r-1)!} x^{(0)}(i), \quad r \in \mathbb{Z}^+,\ k = 1, 2, \dots, n$

When $r \in \mathbb{N}$, $X^{(r)}$ is called an integer-order accumulation sequence; when $r \in \mathbb{R}^+ \setminus \mathbb{N}$, $X^{(r)}$ is called a fractional-order accumulation generation sequence.

Definition 2.2 (Gamma Function)**.** *Let* $n \in \mathbb{R}$ *and* $n \notin \{0, -1, -2, \dots\}$*. The Gamma function* $\Gamma(n)$ *is defined as:*

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt \quad \Gamma(n+1) = n\Gamma(n)$$

The fractional-order accumulation in can be expressed as:

$$x^{(r)}(k) = \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i), \quad r \in \mathbb{Z}^+,\ k = 1, 2, \dots, n$$

Particularly, when $r \in \mathbb{Z}^+$, the expansion coefficient becomes:

$$a_k = \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} = \binom{r+k-i-1}{r-1}$$

Definition 2.3 (Fractional Reducing Generation Operator (r-RGO))**.** *Let* $X^{(0)}$ *be the original sequence. Then, the r-order reducing generation operator (r-RGO) sequence* $X^{(-r)} = \left\{x^{(-r)}(1), x^{(-r)}(2), \dots, x^{(-r)}(n)\right\}$*, for* $r \in \mathbb{R}^+$*, is given as:*

$$x^{(-r)}(k) = \sum_{i=1}^{k-1} \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} x^{(0)}(k-i)$$

The reducing operator is the inverse process of the accumulation operator.

2.1 Derivation of The Fractional Grey Model (FFGM(1,1,i)) with Fiscal drag

Let the original data series be defined as: $x^{(0)} = \left\{x^{(0)}(1), x^{(0)}(2), \dots, x^{(0)}(n)\right\}$, $u^{(0)} = \left\{u^{(0)}(1), u^{(0)}(2), \dots, u^{(0)}(n)\right\}$ where $x^{(0)}$ is the dependent variable and $u^{(0)}$ is the independent (input) variable.

For $r \in (0, 1]$, the r-AGO sequences are:
$x^{(r)}(k) = \sum_{i=1}^{k} \binom{k-i+r-1}{k-i} x^{(0)}(i), \quad u^{(r)}(k) = \sum_{i=1}^{k} \binom{k-i+r-1}{k-i} u^{(0)}(i), \quad k = 1, 2, \dots, n$
where the generalized binomial coefficient is:
$\binom{k-i+r-1}{k-i} = \frac{\Gamma(k-i+r)}{\Gamma(r)\Gamma(k-i+1)}$

$$z^{(r)}(k) = \frac{1}{2}\left(x^{(r)}(k) + x^{(r)}(k-1)\right), \quad k = 2, 3, \dots, n$$

Apply the first-order accumulated generating operation (1-AGO):

$$x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i), \quad u^{(1)}(k) = \sum_{i=1}^{k} u^{(0)}(i), \quad k = 1, 2, \dots, n$$

Define the background value sequence as:

$$z^{(1)}(k) = \frac{1}{2}\left(x^{(1)}(k) + x^{(1)}(k-1)\right), \quad k = 2, 3, \dots, n$$

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Construct the non-homogeneous grey differential equation:

$$\frac{dx^{(r)}(t)}{dt} + ax^{(r)}(t) = b + cu^{(r)}(t). \tag{1}$$

Integrate both sides from $t = k-1$ to $t = k$:

$$\int_{k-1}^{k} \frac{dx^{(r)}(t)}{dt} dt + a\int_{k-1}^{k} x^{(r)}(t)dt = \int_{k-1}^{k} \left(b + cu^{(r)}(t)\right) dt. \tag{2}$$

The first integral is exact:

$$\int_{k-1}^{k} \frac{dx^{(r)}(t)}{dt} dt = x^{(r)}(k) - x^{(r)}(k-1) = x^{(0)}(k). \tag{3}$$

The second integral is approximated using the trapezoidal rule:

$$\int_{k-1}^{k} x^{(r)}(t)dt \approx \frac{x^{(r)}(k) + x^{(r)}(k-1)}{2} = z^{(r)}(k). \tag{4}$$

Assuming $b + cu^{(r)}(t)$ is constant over $[k-1, k]$:

$$\int_{k-1}^{k} \left(b + cu^{(r)}(t)\right) dt = b + cu^{(r)}(k). \tag{5}$$

Substituting the evaluated integrals back:

$$x^{(0)}(k) + az^{(r)}(k) = b + cu^{(r)}(k). \tag{6}$$

The discrete-time grey differential equation is:

$$x^{(0)}(k) + az^{(r)}(k) = b + cu^{(r)}(k), \quad k = 2, 3, \ldots, n$$

Rewriting:

$$x^{(0)}(k) = -az^{(r)}(k) + cu^{(r)}(k) + b$$

We further solve for a,b,c using least square method. Let the matrices be:

$$Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}, \quad B = \begin{bmatrix} -z^{(r)}(2) & u^{(r)}(2) & 1 \\ -z^{(r)}(3) & u^{(r)}(3) & 1 \\ \vdots & \vdots & \vdots \\ -z^{(r)}(n) & u^{(r)}(n) & 1 \end{bmatrix}$$

The discrete-time grey differential equation is:

$$x^{(0)}(k) = -az^{(r)}(k) + cu^{(r)}(k) + b, \quad k = 2, 3, \ldots, n. \tag{7}$$

Define the system matrices, from equation (8)

$$\boldsymbol{Y} = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -z^{(r)}(2) & u^{(r)}(2) & 1 \\ -z^{(r)}(3) & u^{(r)}(3) & 1 \\ \vdots & \vdots & \vdots \\ -z^{(r)}(n) & u^{(r)}(n) & 1 \end{bmatrix}, \boldsymbol{\theta} = \begin{bmatrix} a \\ c \\ b \end{bmatrix}$$

The system can be written as:

$$\boldsymbol{Y} = \boldsymbol{B\theta} \tag{8}$$

The least squares method minimizes the sum of squared errors:

$$J(\boldsymbol{\theta}) = \|\boldsymbol{Y} - \boldsymbol{B\theta}\|^2$$

Taking the derivative with respect to $\boldsymbol{\theta}$ and setting to zero:

$$\frac{\partial J}{\partial \boldsymbol{\theta}} = -2\boldsymbol{B}^T(\boldsymbol{Y} - \boldsymbol{B\theta}) = \boldsymbol{0}$$

This yields the normal equations:

$$\boldsymbol{B}^T\boldsymbol{B\theta} = \boldsymbol{B}^T\boldsymbol{Y} \tag{9}$$

Assuming that $\boldsymbol{B}^T\boldsymbol{B}$ is invertible, the parameter estimate is:

$$\boldsymbol{\theta} = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{Y} \tag{10}$$

The parameter vector is therefore:

$$\begin{bmatrix} a \\ c \\ b \end{bmatrix} = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{Y}$$

where:

- $\boldsymbol{B}^T\boldsymbol{B}$ is a 3×3 matrix of sums:
$\boldsymbol{B}^T\boldsymbol{B} = \begin{bmatrix} \sum[z^{(r)}(k)]^2 & -\sum z^{(r)}(k)u^{(r)}(k) & -\sum z^{(r)}(k) \\ -\sum z^{(r)}(k)u^{(1)}(k) & \sum[u^{(r)}(k)]^2 & \sum u^{(r)}(k) \\ -\sum z^{(r)}(k) & \sum u^{(r)}(k) & n-1 \end{bmatrix}$
- $\boldsymbol{B}^T\boldsymbol{Y}$ is:

$$\boldsymbol{B}^T\boldsymbol{Y} = \begin{bmatrix} -\sum z^{(r)}(k)x^{(0)}(k) \\ \sum u^{(r)}(k)x^{(0)}(k) \\ \sum x^{(0)}(k) \end{bmatrix}$$

Estimate the parameters using least squares:

$$\begin{bmatrix} a \\ c \\ b \end{bmatrix} = (B^TB)^{-1}B^TY$$

Solve the differential equation:

$$x^{(r)}(k) = \left(x^{(0)}(1) - \frac{b + cu^{(r)}(k)}{a}\right) e^{-a(k-1)} + \frac{b + cu^{(r)}(k)}{a}$$

Apply the inverse accumulated generating operation (r-AGO) to forecast the original values:

$$x^{(0)}(k) = x^{(r)}(k) - x^{(r)}(k-1)$$

2.2 Deriving The Fiscal drag function

We are modeling how tax revenue potential from mid-level Nigerian civil servants (CONPSS 7–10) decays over time due to under utilization of state-subsidized human capital, as education is not just a path to personal upliftment but ——. The government subsidizes OND holders, BSc graduates, and professionals CONPSS 7, 8, and 10 in public institutions, subsidizing their journey to upward mobility. Meanwhile, appointment-based officers at CONPSS 15 contribute far more, not necessarily because of higher productivity, but due to positional privilege because of this We design a tax rate function for civil service bands (CONPSS 7,8,10). To analytically capture this effect, we introduce a tax decay function that represents the diminishing tax potential of an employee over time:

$$cu^r(N_i) = N_i \cdot \exp^{(-\lambda t)} \tag{11}$$

Where:

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- N_i = The salary at level i, or more precisely, the expected salary at the next promotion level
- λ = The skill depreciation coefficient — a measure of how quickly unused or underutilized skills lose economic value
- t = The allocated time (in months) for the employee to acquire a new skill

Then the Fractional Grey Model becomes ($FFGM(1,1,u^r(N_i))$) as $\lambda_1 > \lambda_2 > \lambda_3$ these represent slow, moderate, and fast skill decay, respectively. some key differences between the GM(1,1) and the FFGM is highlighted

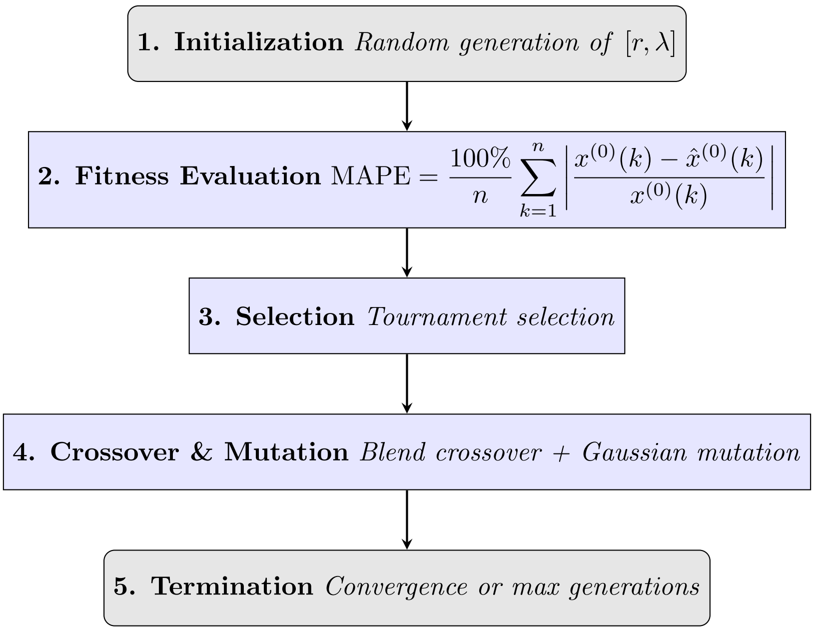
Key Differences Between GM(1,1) and FFGM

Feature	**GM(1,1)**	**FFGM**
Accumulation	First-order ($r = 1$)	Fractional-order ($0 < r \leq 2$)
Equation	$\frac{dx^{(1)}}{dt} + ax^{(1)} = b$	$\frac{dX^{(r)}}{dt} + aX^{(r)} = be^{-\lambda t}$
Solution	Pure exponential	Exponential + fiscal drag decay
Flexibility	Less adaptive	Better for nonlinear trends
Parameters	a, b	a, b, r, λ

2.3 Genetic Algorithm Optimization

The parameters r (fractional order) and λ (education dividend coefficient) are optimized using a Genetic Algorithm (GA) to minimize forecasting error. The GA workflow is as follows:

The optimal r and λ are then used in the FFGM(1,1,i) model for forecasting.



Definition 2.4. *Let $X^{(0)}$ be the original sequence from Definition 1, and let $X^{(r)}$ be the r-order accumulation generation sequence of $X^{(0)}$,then the following equation defines the FFGM model:*

$$x_k^{(r-1)} + az_k^{(r)} = bk + N_O e^{-\lambda t_2} \tag{12}$$

where $x_k^{(r)}$ is:

$$x_k^{(r-1)} = x_k^{(r)} - x_{k-1}^{(r)}, \quad z_k^{(r)} = \frac{1}{2}\left(x_k^{(r)} + x_{k-1}^{(r)}\right) \tag{13}$$

We define an exponential decay function $u_k^{(r)}$ and its associated forms analogous to $x_k^{(r)}$ and $z_k^{(r)}$ in the FSIGM model:

$$u_k^{(r)} = N_O \cdot e^{-\lambda k} \tag{14}$$

$$u_k^{(r-1)} = u_k^{(r)} - u_{k-1}^{(r)} = N_O \left(e^{-\lambda k} - e^{-\lambda(k-1)}\right) \tag{15}$$

$$v_k^{(r)} = \frac{1}{2}\left(u_k^{(r)} + u_{k-1}^{(r)}\right) = \frac{N_O}{2}\left(e^{-\lambda k} + e^{-\lambda(k-1)}\right) \tag{16}$$

Specifically, when $r = 1$, equation (19) becomes:

$$x_k^{(0)} + az_k^{(1)} = bk + N_O e^{-\lambda t_2} \tag{17}$$

This is the original form of the SIGM model with exponential decay in place of the constant term.

2.4 Property of the FFGM model

The parameter vector of the FSIGM model $\hat{a} = [a, b, c]^T$, using least squares estimation, is given by:

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$$\hat{a} = (B^T B)^{-1} B^T Y$$

where

$$B = \begin{bmatrix} -z_{r_2} & N_O e^{-\lambda t_2} & 1 \\ -z_{r_3} & N_O e^{-\lambda t_3} & 1 \\ \vdots & \vdots & \vdots \\ -z_{r_n} & N_O e^{-\lambda t_n} & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} x_{r_2 - 1} \\ x_{r_3 - 1} \\ \vdots \\ x_{r_n - 1} \end{bmatrix}$$

The fractional property can be redefined as a structure of matrix B in the FSIGM model:

$$B = \begin{bmatrix} -z_2^{(r)} & N_O e^{-\lambda \cdot 2} & 1 \\ -z_3^{(r)} & N_O e^{-\lambda \cdot 3} & 1 \\ \vdots & \vdots & \vdots \\ -z_n^{(r)} & N_O e^{-\lambda \cdot n} & 1 \end{bmatrix}$$

Using the definition:

$$z_k^{(r)} = \frac{1}{2}\left(x_k^{(r)} + x_{k-1}^{(r)}\right)$$

we rewrite B as:

$$B = \begin{bmatrix} -\frac{x_1^{(r)} + x_2^{(r)}}{2} & N_O e^{-\lambda \cdot 2} & 1 \\ -\frac{x_2^{(r)} + x_3^{(r)}}{2} & N_O e^{-\lambda \cdot 3} & 1 \\ \vdots & \vdots & \vdots \\ -\frac{x_{n-1}^{(r)} + x_n^{(r)}}{2} & N_O e^{-\lambda \cdot n} & 1 \end{bmatrix}$$

Now express each $x_k^{(r)}$ in terms of the original sequence $x_i^{(0)}$ using the general form of the r-order Accumulated Generating Operation (AGO), based on the Gamma function:

$$x_k^{(r)} = \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x_i^{(0)}$$

Substitute into B. For example:

Row 1 ($k = 2$):

$$-\frac{1}{2} \sum_{i=1}^{2} \frac{\Gamma(r+2-i)}{\Gamma(2-i+1)\Gamma(r)} x_i^{(0)} + \sum_{i=1}^{1} \frac{\Gamma(r+2-i)}{\Gamma(2-i+1)\Gamma(r)} x_i^{(0)}$$

Row 2 ($k = 3$):

$$-\frac{1}{2} \sum_{i=1}^{3} \frac{\Gamma(r+3-i)}{\Gamma(3-i+1)\Gamma(r)} x_i^{(0)} + \sum_{i=1}^{2} \frac{\Gamma(r+3-i)}{\Gamma(3-i+1)\Gamma(r)} x_i^{(0)}$$

General row k:

$$-\frac{1}{2} \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x_i^{(0)} + \sum_{i=1}^{k-1} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x_i^{(0)}$$

So finally, matrix B takes the form:

$$B = \begin{bmatrix} (\text{Gamma form}, k=2) & N_O e^{-\lambda \cdot 2} & 1 \\ (\text{Gamma form}, k=3) & N_O e^{-\lambda \cdot 3} & 1 \\ \vdots & \vdots & \vdots \\ (\text{Gamma form}, k=n) & N_O e^{-\lambda \cdot n} & 1 \end{bmatrix}$$

3 Numerical Analysis

To illustrate the impact of the recently introduced tax rates on the salaries of Nigerian CONPSS (Consolidated Public Service Salary Structure) employees, consider the following example. The table below presents selected salary levels alongside their corresponding tax rates and the resulting tax amounts. Notably, while lower salary levels (Level 7 to Level 10) are taxed at a flat rate of 15%, the target higher salary level (Level 15) attracts an increased tax rate of 18%. This adjustment reflects the government's approach to progressive taxation, where higher income earners contribute a larger proportion of their earnings in taxes.

However, to further enhance government revenue, there is a proposal to implement fiscal drag by scaling the tax rates for Levels 7 and 8 to Levels 10 and above. This means gradually increasing the tax rates for these lower and middle salary bands to better align with inflation and wage growth, enabling the government to generate more tax revenue based on the population size within these levels.

CONPSS Level	**Salary (₦)**	**Tax Rate (%)**	**Tax (₦)**
Level 7	1,500,000	15	225,000
Level 8	1,700,000	15	255,000
Level 10	2,100,000	15	315,000
Target (Level 15)	3,500,000	18	**630,000**

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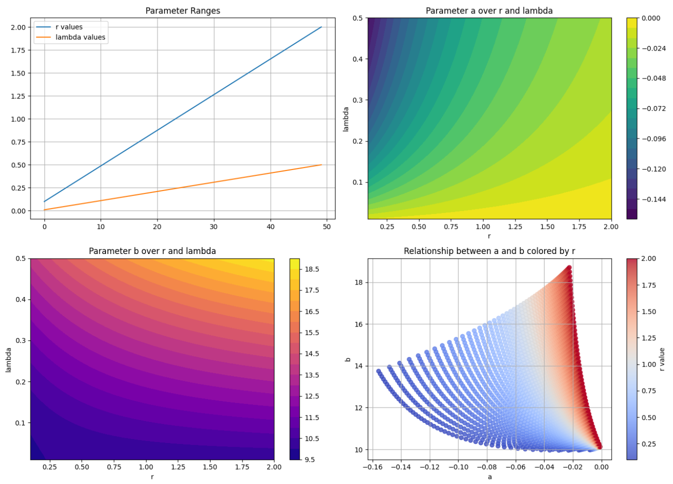

(a) Actual vs. FFGM and GM(1,1) Model Predictions (2)

Figure 3: Comparison of Actual Tax Revenue versus Model Predictions across Two Analyses

4 Conclusion and Recommendations

4.1 Conclusion

This study has demonstrated the efficacy of the Fractional Grey Model with Fiscal Drag (FFGM(1,1,i)) in forecasting tax revenue potential from Nigeria's mid-level civil servants under CONPSS Levels 07-10. By integrating fractional calculus principles with grey system theory, the proposed model successfully captures the complex dynamics of skill depreciation and fiscal drag that characterize this segment of the public workforce.

The numerical analysis reveals that the FFGM model outperforms the traditional GM(1,1) approach, achieving significantly lower prediction errors across multiple forecasting periods. This enhanced accuracy stems from the model's ability to:

- Incorporate fractional-order accumulation for better handling of non-homogeneous exponential sequences
- Account for skill depreciation through the fiscal drag function $u^{(r)}(N_i) = N_i \cdot e^{-\lambda t}$
- Capture long-range dependencies and memory effects in economic time series data
- Adapt to small sample sizes and data irregularities common in public sector contexts

The fiscal drag component specifically addresses the critical issue of underutilized human capital, quantifying how untapped potential translates into lost tax revenue over time. This provides a mathematical foundation for understanding the economic consequences of stagnant career progression and limited skill development opportunities in the civil service.

4.2 Policy Recommendations

Based on the findings of this study, the following policy recommendations are proposed:

4.2.1 Immediate Interventions

1. **Accelerated Promotion Pathways**: Implement structured promotion programs for CONPSS Levels 07-10 employees, reducing the time between grade levels through performance-based advancement rather than strict seniority.
2. **Targeted Skill Development**: Establish government-subsidized training programs focused on digital literacy, administrative efficiency, and technical competencies that align with current economic needs.
3. **Tax Structure Optimization**: Gradually adjust tax rates for mid-level civil servants to reflect their enhanced productivity and income potential, following the fiscal drag principles outlined in the model.

4.2.2 Medium-term Strategies

1. **Human Capital Investment Fund**: Create a dedicated funding mechanism for continuous professional development of mid-level civil servants, with clear metrics for return on investment through increased tax contributions.
2. **Performance-Linked Compensation**: Introduce bonus structures and productivity incentives that reward skill acquisition and application, creating direct financial motivation for professional growth.
3. **Cross-Training Programs**: Develop rotational assignments and inter-ministerial training opportunities to broaden experience and enhance workforce flexibility.

4.2.3 Long-term Institutional Reforms

1. **Data-Driven Workforce Planning**: Establish a comprehensive database tracking career progression, skill development, and productivity metrics to support evidence-based policy decisions.
2. **Predictive Analytics Integration**: Incorporate advanced forecasting models like FFGM into national budget planning and revenue projection processes.

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3. **Public-Private Skill Partnerships**: Foster collaborations between government agencies and private sector organizations for knowledge transfer and contemporary skill development.

4.3 Future Research Directions

This study opens several avenues for further investigation:

- Extension of the FFGM framework to incorporate multiple external variables affecting civil service productivity
- Application of the model to other developing economies with similar public sector structures
- Integration of machine learning techniques with fractional grey models for enhanced predictive accuracy
- Longitudinal studies tracking the actual fiscal impact of implemented skill enhancement programs

The FFGM(1,1,i) model provides not just a forecasting tool, but a comprehensive framework for understanding and optimizing the relationship between human capital investment and fiscal returns in the public sector. By adopting the recommendations outlined above, the Nigerian government can transform its mid-level civil service from a cost center into a dynamic engine of economic growth and sustainable revenue generation.

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