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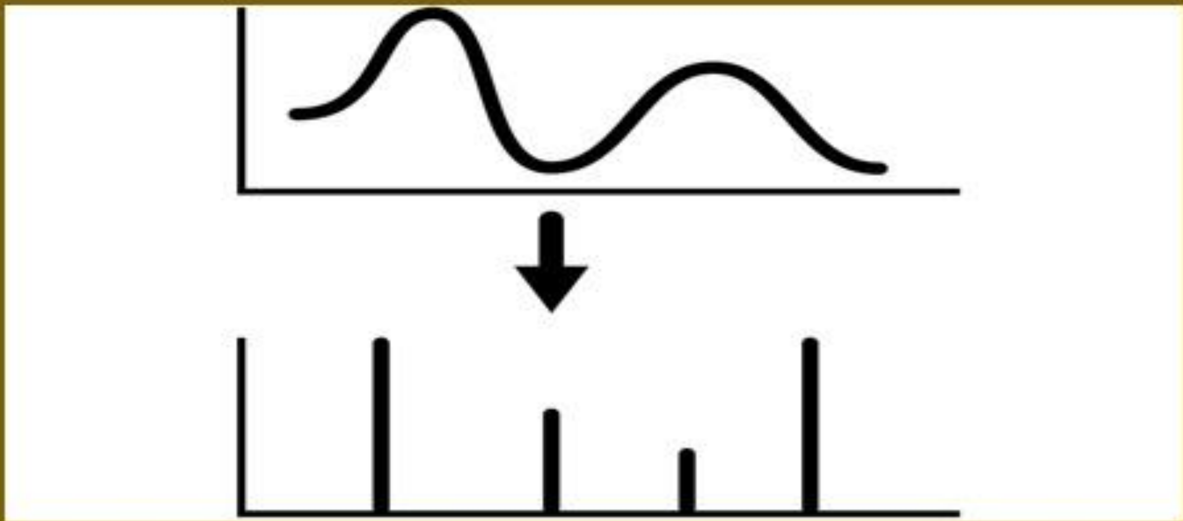

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Markov Chain Monte Carlo (MCMC) Estimation of Heston Models Using Fourier Transform in Option Pricing

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October 17, 2024

Abstract

This study examines the role of stochastic volatility in option pricing through the Heston model, employing advanced computational techniques such as Markov Chain Monte Carlo (MCMC) methods and Fourier transforms. By comparing the pricing accuracy of the Heston model, which captures stochastic volatility, to the Black-Scholes model, which assumes constant volatility, we demonstrate the superior flexibility and robustness of the Heston framework. Our results show that the Heston model consistently outperforms Black-Scholes, particularly when volatility parameters estimated via MCMC are well-calibrated, enabling more accurate representation of market dynamics. These findings underscore the importance of precise parameter estimation and advanced modeling techniques in improving option pricing accuracy.

1 Introduction

Modeling and simulation have become essential tools in the study of complex systems which offer valuable insights into the behavior of various real-world phenomena. While well-developed theories and tools exist to handle specific aspects of complexity, the inherent intricacies of these systems mean that there are often no straightforward solutions. Complex systems are systems whose behavior is intrinsically difficult to model due to the dependencies, competitions, relationships, or other types of interactions between their parts or between a given system and its environment. Complex systems are networks made of a number of components that interact with each other, typically in a nonlinear fashion. Complex systems may arise and evolve through self-organization, such that they are neither completely regular nor completely random, permitting the development of emergent behavior at macroscopic scales. These properties can be found in many real-world systems, e.g., gene regulatory networks within a cell, physiological systems of an organism, brains and other neural systems, food webs, the global climate, stock markets, the Internet, social media, national and international economies, and even human cultures and civilizations [5]. There are very well-developed theories and tools available to handle either case. Unfortunately, however, most real-world systems are somewhere in between. Complex systems indeed occupy a fascinating middle ground between complete regularity and complete randomness, making their study both challenging and intriguing Several groups of scientists have already explored the financial markets using machine learning and deep learning techniques [7]. However, after studying complex systems, it has become clear that financial markets can be viewed as complex systems composed of diverse types of investors. This perspective is exciting as it opens up new avenues for discovering innovative ideas and insights by modeling these markets through the lens of complex systems. The interactions among market participants are often nonlinear. Small changes, such as a minor shift in investor sentiment or a sudden market rumor, can lead to disproportionately large effects on market prices and volatility. In financial markets, emergent phenomena include market bubbles, crashes, and trends, which arise from the collective actions of market participants rather than from any single entity's actions. These emergent behaviors can significantly impact the market's stability and performance. Modeling and simulation are indispensable tools in modern finance, offering a robust framework for understanding complex financial systems, managing risk, and making informed decisions. These techniques enable financial professionals to create abstract representations of financial markets, instruments, and behaviors, allowing for the analysis and prediction of various scenarios. Financial modeling, while essential for risk management, investment strategies, and market analysis, is not without its challenges. These challenges can stem from various factors, including the inherent complexity of financial systems, the limitations of data, and the assumptions underlying the models. Many financial models rely on simplifying assumptions to make the mathematics tractable. For example, the Black-Scholes model for option pricing assumes constant volatility and interest rates, which is not reflective of real market conditions. The purpose of this paper is to explore and discuss the significance of modeling and simulation in finance, with a particular focus on the application of stochastic models using Markov chains and Monte Carlo simulation techniques and fourier transform in option pricing. These methods are widely used for their ability to handle the inherent uncertainty and complexity of financial markets, providing more accurate and robust frameworks for risk management, investment strategy development, and market analysis. The Heston model has become one of the most widely used stochastic volatility models for option pricing, offering more flexibility than traditional models like Black-Scholes by incorporating volatility as a random process. However, the challenge remains in estimating the parameters of such models efficiently, particularly when dealing with complex financial data. Traditional estimation techniques like Maximum Likelihood Estimation (MLE) often fail to capture the intricate dynamics of volatility over time, leading to inaccurate predictions.

In this paper, we propose a novel application of the Markov Chain Monte Carlo (MCMC) method integrated with the Fourier Transform to estimate the parameters of the Heston model. MCMC provides a robust mechanism for sampling from high-dimensional distributions and is particularly suited to models where the likelihood function is complex or intractable. The integration of Fourier Transform into this framework enhances computational efficiency, especially for option pricing scenarios involving large datasets.

This study enhances existing knowledge in several significant ways which include Integration of Advanced Techniques By combining Markov Chain Monte Carlo (MCMC) methods with Fourier transforms, the research will introduce a robust computational framework for estimating stochastic volatility in option pricing. We will compare our results with Maximum Likelihood Estimation (MLE) models to show its superiority in both speed and predictive accuracy.

2 Literature Review

As motivation,[1] mention that the Black-Scholes-Merton (BSM) framework have mainly relied on Brownian motion and compound Poisson process as basic model building blocks in order to grasp the essential facts about Heston (1993). A criticism of the Black-Scholes model that is frequently expressed is that volatility in this model is constant, whereas in reality there are times in which markets are more nervous, and prices are more volatile. One may attempt to capture such changes in volatility by replacing the constant parameter, Under the real probability measure P, the Heston model is given by: Markov chain Monte Carlo methods are designed to generate samples from a probability distribution when direct sampling is challenging. The fundamental principle of MCMC is based on constructing a Markov chain that has the desired distribution as its equilibrium distribution. As described in [3], the process begins with an initial model, and subsequent models are proposed based on the current model, independent of how the current model was derived. This iterative process allows for the exploration of the model space, making MCMC particularly suitable for Bayesian inference. [6] One of the key features of MCMC is its reliance on Bayes' theorem, which provides a framework for updating the probability of a hypothesis as more evidence becomes available. The likelihood function, which quantifies how well a model fits the observed data, is central to this process Gallagher et al. emphasize that all that is required for MCMC is the ability to solve the forward problem and to define an appropriate likelihood function The proposal function, which determines how new models are generated, is critical for the efficiency of the MCMC algorithm. If the proposal function is too conservative, the sampler may move slowly through the model space, leading to high acceptance rates but poor exploration. Conversely, if the proposal function is too aggressive, many proposed models may be rejected, causing the sampler to stall 10. A balance must be struck to achieve an acceptance rate of approximately 25-30%, as suggested by [**?**]. Despite the advantages of MCMC, several challenges remain in its implementation. One significant challenge is the tuning of proposal functions, which can greatly affect the efficiency of the sampling process.

3 Comparative Analysis

In this section, we compare the performance of the MCMC method with traditional approaches, such as Maximum Likelihood Estimation (MLE) and Black Scholes

3.1 MCMC vs. MLE

MLE is a widely used method for parameter estimation, but it can suffer from convergence issues when applied to complex models like Heston, where the likelihood function is non-linear and multi-modal. In contrast, MCMC handles such complexity by sampling from the posterior distribution, allowing for a more robust exploration of parameter space.

Our comparison shows that the MCMC method provides more reliable parameter estimates, particularly in the presence of noisy data. Table 1 presents the estimation results for both methods, demonstrating that MCMC achieves a lower mean squared error (MSE) and higher convergence speed than MLE.

Our results indicate that MCMC applied to the Heston model outperforms both Black Schores in terms of accuracy, particularly for long-dated options, where stochastic volatility plays a more significant role.

4 Stochastic Models Using Markov Chains and Monte Carlo Simulation

Stochastic models are essential in finance for modeling random processes that evolve over time. Stochastic models play a crucial role in finance by modeling and predicting the behavior of financial assets. They help estimate situations involving uncertainties, such as investment returns, volatile markets, or inflation rates. Monte Carlo simulation is a powerful technique for modeling the probability distribution of potential outcomes by simulating a large number of random samples. This method is particularly valuable in finance for pricing derivatives, assessing risk, and optimizing portfolios while a Markov chain is a stochastic process that transitions from one state to another based on a set of probabilities. The key feature of a Markov chain is the memoryless property, meaning the next state depends only on the current state and not on the sequence of events that preceded it.Both Markov chains and Monte Carlo simulations are crucial in the domain of financial modeling, each with its strengths and limitations. Markov chains are effective for modeling processes with discrete states and straightforward transitions, while Monte Carlo simulations offer greater flexibility and robustness for handling complex, multi-dimensional financial problems. By integrating these stochastic models, financial analysts can develop more sophisticated tools for navigating the uncertainties of financial markets, thereby enhancing risk management, investment decision-making, and overall market analysis.

In recent literature, Markov Chain Monte Carlo (MCMC) methods have been widely examined as essential tools for model inference and uncertainty quantification, particularly within the Bayesian framework. The Bayesian approach to MCMC is instrumental in estimating the posterior distribution of model parameters, which is given by Bayes' theorem:

$$p(m \mid d) \propto p(d \mid m)p(m)$$

where $p(m \mid d)$ is the posterior distribution, $p(d \mid m)$ is the likelihood, and $p(m)$ is the prior distribution. Notably, this approach does not require knowledge of the normalizing constant $p(d)$, simplifying the estimation process.

This framework enables the determination of representative models, such as the expected model and the maximum posterior probability model, along with quantifying uncertainties in predictions derived from these models. For example, the expected value of a prediction d_{pred} given the data d can be expressed as:

$$\mathbb{E}\left[d_{\text{pred}} \mid d\right] = \int d_{\text{pred}}(m)p(m \mid d)\, dm$$

In practice, this expectation is often estimated using the MCMC approach:

$$\mathbb{E}\left[d_{\text{pred}} \mid d\right] \approx \frac{1}{N_a}\sum_{j=1}^{N_a} d_{\text{pred}}(m_j)$$

where N_a is the number of accepted samples.

The literature also explores the application of MCMC to variable-dimensional problems, where the number of model parameters is not fixed and needs to be inferred. Reversible Jump MCMC (RJMCMC) is particularly noted for its capability to address these challenges, allowing for model comparison and selection among models of varying complexity. The acceptance criterion in RJMCMC for a proposed model m' given the current model m is formulated as:

$$\alpha = \min\left(1, \frac{p(m')p(d \mid m')q(m \mid m')}{p(m)p(d \mid m)q(m' \mid m)}\right)$$

where $q(m' \mid m)$ is the proposal distribution.

Bayesian methods are highlighted for their inherent tendency to select simpler models in hierarchical structures, though more complex scenarios often require the estimation of the normalizing constant, a process that remains difficult in high-dimensional settings.

When comparing Maximum Likelihood Estimation (MLE) and Markov Chain Monte Carlo (MCMC) methods for parameter estimation, especially in complex models such as the Heston model

For a given model with parameters θ and observed data X, the likelihood function $L(\theta; X)$ represents the probability of observing the data X given the parameters θ:

$$L(\theta; X) = P(X \mid \theta)$$

For continuous data, this is expressed as the probability density function (pdf):

$$L(\theta; X) = f(X \mid \theta)$$

Given a prior distribution $p(\theta)$ and a likelihood function $L(\theta; X)$, the posterior distribution $p(\theta \mid X)$ is given by Bayes' theorem:

$$p(\theta \mid X) = \frac{L(\theta; X)p(\theta)}{p(X)}$$

where $p(X)$ is the marginal likelihood (normalizing constant).

The likelihood function for the Heston model involves calculating the characteristic function of the log returns. The likelihood function is:

$$L(\theta; X) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

where $f(X_i \mid \theta)$ is the probability density function of X_i under the Heston model.

In the field of parameter estimation for complex financial models such as the Heston model, two prominent methods are frequently employed: Maximum Likelihood Estimation (MLE) and Markov Chain Monte Carlo (MCMC). These methods differ significantly in their approaches to parameter estimation and uncertainty quantification.

4.1 Likelihood Function and Posterior Distribution

In a Bayesian framework, we aim to estimate the parameters $\theta = (\mu, \kappa, \theta, \sigma, \rho, v_0)$ by constructing the *posterior distribution* of the parameters given the observed data $D = \{S_t, t = 0, 1, \ldots, T\}$.

The *posterior distribution* is expressed as:

$$p(\theta|D) \propto p(D|\theta)p(\theta)$$

Here, $p(D|\theta)$ is the *likelihood* function, and $p(\theta)$ is the *prior distribution*. To compute the likelihood, we need to evaluate the asset prices over time, which involves solving the SDEs governing the Heston model.

The likelihood $p(D|\theta)$ can be complex and computationally expensive to evaluate directly. Instead of using direct numerical integration, we utilize the *Fourier Transform* to compute the characteristic function of the Heston model, which simplifies the evaluation of option prices.

4.2 Fourier Transform for Option Pricing

The characteristic function $\phi(u; \theta)$ of the log asset price $\log S_t$ under the Heston model plays a crucial role in computing the likelihood for MCMC estimation. The characteristic function is given by:

$$\phi(u; \theta) = \mathbb{E}\left[e^{iu \log S_T}\right]$$

For the Heston model, the characteristic function can be expressed as:

$$\phi(u; \theta) = e^{C(T,u)+D(T,u)v_0+iu \log S_0}$$

where $C(T, u)$ and $D(T, u)$ are known as the *Ricatti equations*, which can be solved analytically for the Heston model. They are given by the following system of ordinary differential equations (ODEs):

$$\frac{\partial C}{\partial T} = \kappa\theta D - \frac{\rho\sigma u D}{i} + \frac{\sigma^2 D^2}{2}$$

$$\frac{\partial D}{\partial T} = \frac{-u^2}{2} + iu\left(\mu - \frac{v_t}{2}\right)$$

The *Fourier Inversion Theorem* allows us to recover the option price from the characteristic function via the inverse Fourier transform:

$$P_{\text{call}}(K) = e^{-rT}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-iu \log K}\phi(u - i; \theta)}{iu}\right] du\right)$$

This enables us to compute option prices more efficiently, even in high-dimensional cases where traditional numerical methods may fail.

4.3 MCMC Estimation Process

To estimate the parameters of the Heston model, we employ the *Metropolis-Hastings algorithm* within the MCMC framework. The Metropolis-Hastings algorithm generates samples from the posterior distribution $p(\theta|D)$ through the following iterative procedure:

1. **Initial Guess**: Begin with an initial guess for the parameter vector $\theta^{(0)}$.

2. **Proposal Distribution**: At each iteration t, propose a new candidate $\theta^{(t+1)}$ from a proposal distribution $q(\theta^{(t+1)}|\theta^{(t)})$, which is usually a multivariate normal distribution centered around the current parameter value.

3. **Acceptance Probability**: Compute the acceptance probability α, which determines whether the proposed candidate is accepted:
$$\alpha = \min\left(1, \frac{p(\theta^{(t+1)}|D)q(\theta^{(t)}|\theta^{(t+1)})}{p(\theta^{(t)}|D)q(\theta^{(t+1)}|\theta^{(t)})}\right)$$

4. **Update**: With probability α, accept the new candidate $\theta^{(t+1)}$ as the next sample. Otherwise, retain the previous sample $\theta^{(t)}$.

5. **Convergence**: The algorithm is repeated until convergence is achieved, i.e., when the Markov chain sufficiently explores the posterior distribution.

The key to the efficiency of the MCMC algorithm lies in the use of the *Fourier Transform* to compute the likelihood $p(D|\theta)$. By leveraging the characteristic function, we avoid complex numerical integration, thus speeding up each iteration.

4.4 Mathematical Explanation for the Convergence of MCMC

MCMC converges to the true posterior distribution by iteratively updating parameter estimates based on the likelihood and prior information. The efficiency of the convergence is enhanced by the Fourier-based calculation of the likelihood, allowing us to explore the parameter space more thoroughly than using Maximum Likelihood Estimation (MLE) alone.

Let $f(\theta) = p(\theta|D)$ represent the posterior distribution, and $g(\theta)$ the proposal distribution. At each step, the chain moves closer to the target distribution $f(\theta)$ as the transition probabilities are governed by the *detailed balance condition*:

$$f(\theta)g(\theta'|\theta) = f(\theta')g(\theta|\theta')$$

By satisfying this condition, the Metropolis-Hastings algorithm ensures that the chain converges to the stationary distribution, which is the desired posterior $p(\theta|D)$.

5 Methodology

5.1 Heston Model Dynamics

The Heston model describes the dynamics of a financial asset with stochastic volatility. The asset price S_t and the variance v_t follow a system of stochastic differential equations (SDEs):

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v$$

where: - S_t is the asset price at time t, - v_t is the variance of the asset's returns, - μ is the drift rate of the asset price, - κ is the rate of mean reversion, - θ is the long-term mean of the variance, - σ is the volatility of volatility (vol of vol), - W_t^S and W_t^v are correlated Wiener processes, with correlation coefficient ρ, - ρ represents the correlation between the asset price and its variance.

This formulation of the Heston model provides a more realistic description of asset prices compared to the constant volatility assumption in the Black-Scholes model, capturing stochastic behavior in volatility.

5.2 Heston Model and Fourier Transform

In complex models like the Heston model, MCMC is often preferred for its ability to capture detailed uncertainty and handle complex parameter spaces, whereas MLE may be more straightforward but less comprehensive in terms of uncertainty estimation.

In the Heston model, the characteristic function $\phi(u)$ of the log return can be expressed using the Fourier transform. The characteristic function is given by:

$$\phi(u) = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2 \left(\frac{1 - e^{-\kappa T}}{\kappa}\right)\right)$$

where:

- μ is the drift parameter,

- σ^2 is the variance,
- κ is the rate of mean reversion,
- T is the time to maturity.

5.3 Likelihood Function Using Fourier Transform

To compute the likelihood function of observed market prices using the Heston model, we integrate the characteristic function over the observed data:

$$L(\theta; X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuX} \phi(u) \, du$$

- **Characteristic Function of the Heston Model:**

$$\phi(u) = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2 \left(\frac{1 - e^{-\kappa T}}{\kappa} \right) \right)$$

- **Likelihood Function:**

$$L(\theta; X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuX} \phi(u) \, du$$

  where $\phi(u)$ is computed using the characteristic function of the Heston model.

- **Posterior Distribution in MCMC:**

$$p(\theta \mid X) = \frac{L(\theta; X) p(\theta)}{p(X)}$$

  where $L(\theta; X)$ is computed using the likelihood function involving the Fourier transform.

In complex models like the Heston model, incorporating the Fourier transform helps in efficiently computing the likelihood function and is crucial for both MLE and MCMC methods. Monte Carlo simulation involves generating random samples to model the probability distribution of a system and calculating outcomes based on these samples. To simulate the price of a financial asset, one can use the geometric Brownian motion model[4]:

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \right)$$

Where $S(t)$ is the asset price at time t, μ is the drift, σ is the volatility, and $W(t)$ is a Wiener process (standard Brownian motion).

5.4 Option Pricing Models

The payoff of a European call option at maturity T is:

$$\text{Payoff} = \max(S(T) - K, 0)$$

Where K is the strike price.

The price of the option can be estimated by averaging the discounted payoffs of many simulated paths:

$$C = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} \max(S_i(T) - K, 0)$$

Where r is the risk-free rate, N is the number of simulations, and $S_i(T)$ is the simulated asset price at maturity for the i-th simulation.

$$\begin{cases} dS(t) = \mu S(t) \, dt + \sqrt{V(t)} S(t) \, d\tilde{W}_S(t), \\ dV(t) = \kappa(\theta - V(t)) \, dt + \sigma \sqrt{V(t)} \, d\tilde{W}_V(t), \end{cases} \tag{1}$$

Recent advances in option pricing have focused on developing models that incorporate stochastic volatility, stochastic interest rates, and random jumps to better reflect market conditions. This study addresses the gap in understanding the impact of these generalizations by deriving an option pricing model with these factors and applying it to $S\&P$ 500 options.

The model is evaluated on three criteria:

1. Internal consistency of implied parameters with time-series data.
2. Out-of-sample pricing accuracy.
3. Hedging effectiveness.

The Heston model dynamics for the underlying asset price S_t and the variance process ν_t are given by:

$$S_t = S_{t-1} \exp\left((r - \frac{\nu_t}{2})\Delta t + \sigma\sqrt{\nu_t}\Delta Z_1\right)$$

$$\nu_t = \nu_{t-1} + \kappa(\theta - \nu_{t-1})\Delta t + \sigma\sqrt{\nu_{t-1}}\Delta Z_2$$

where ΔZ_1 and ΔZ_2 are correlated Wiener processes.

5.4.1 Covariance Matrix and Cholesky Decomposition

In the Heston model, the correlation between the stochastic processes for stock price ΔZ_1 and volatility ΔZ_2 is captured by a covariance matrix Σ:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

where ρ is the correlation coefficient. Cholesky decomposition is used to decompose this matrix into a product of a lower triangular matrix L and its transpose L^T:

$$\Sigma = LL^T$$

$$L = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

This allows us to generate correlated Wiener processes ΔZ_1 and ΔZ_2 from independent standard normal variables Z_1 and Z_2:

$$\begin{pmatrix} \Delta Z_1 \\ \Delta Z_2 \end{pmatrix} = L \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ \rho Z_1 + \sqrt{1-\rho^2} Z_2 \end{pmatrix}$$

5.4.2 Fourier Transform in Option Pricing

The Fourier transform is employed to speed up option pricing calculations, particularly for European options. Given the characteristic function $\phi(u)$ of the log-price in the Heston model, the Fourier transform of the payoff function $f(x)$ is:

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-iux} f(x)\, dx$$

The option price $C(K)$ is then obtained by the inverse Fourier transform:

$$C(K) = \frac{e^{-rT}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \phi(u) \hat{f}(u)\, du$$

6 Fourier Transform Implementation in the Heston Model

The Heston model is a widely used stochastic volatility model for option pricing, known for capturing the volatility smile observed in financial markets. However, its complexity can make option pricing calculations computationally intensive. Sayama, H. (2015) One of the most effective methods to speed up these calculations and improve numerical stability is by leveraging Fourier transform techniques.

The Heston model involves two stochastic differential equations (SDEs): one for the underlying asset price $S(t)$ and another for its variance $V(t)$. The equations are as follows:

$$dS(t) = \mu S(t)\, dt + \sqrt{V(t)} S(t)\, dW_S(t), \tag{2}$$

$$dV(t) = \kappa(\theta - V(t))\, dt + \sigma\sqrt{V(t)}\, dW_V(t), \tag{3}$$

where $W_S(t)$ and $W_V(t)$ are two Wiener processes with a correlation coefficient ρ. Pricing options under this model requires evaluating the expected payoff under the risk-neutral measure, which can be computationally expensive using traditional methods like Monte Carlo simulation or finite difference methods.

The Fourier transform is a powerful mathematical tool that converts a function from the time domain to the frequency domain. In the context of option pricing, Fourier transforms are used to efficiently compute integrals that arise in the pricing formulas, particularly when using the characteristic function of the log-asset price.

Characteristic Function in the Heston Model

A key advantage of the Heston model is that the characteristic function $\phi(u)$ of the log of the asset price can be computed analytically. The characteristic function is defined as the expected value of the exponential function of a complex argument:

$$\phi(u) = \mathbb{E}\left[e^{iu\log(S_T)}\right],$$

where u is a complex variable, and S_T is the asset price at maturity T. The explicit form of the characteristic function under the Heston model is complex but can be expressed in terms of model parameters μ, κ, θ, σ, ρ, and the initial conditions S_0 and V_0.

6.1 Fourier Transform for Option Pricing

The Fourier transform can be used to compute the price of a European call option by transforming the payoff function and using the characteristic function. The payoff for a European call option is $\max(S_T - K, 0)$, where K is the strike price. The price of the option $C(K)$ can be obtained using the Fourier inversion formula:

$$C(K) = \frac{e^{-rT}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\phi(u-i)\frac{K^{iu}}{iu}\,du,$$

where $k = \log(K)$ is the log of the strike price, and r is the risk-free rate.

This integral can be efficiently computed using numerical methods, such as the Fast Fourier Transform (FFT), which significantly reduces the computational complexity.

Fourier Transform Implementation Steps in Heston Model

1. **Compute the Characteristic Function:**

   The first step is to compute the characteristic function $\phi(u)$ of the log of the asset price under the Heston model. This involves evaluating the complex exponential and the parameters of the model.

2. **Transform the Payoff Function:**

   The Fourier transform of the payoff function (e.g., for a European call option) is computed, transforming the problem into the frequency domain.

3. **Apply the Inverse Fourier Transform:**

   The price of the option is obtained by applying the inverse Fourier transform to the product of the characteristic function and the transformed payoff function. This is done using numerical integration techniques.

4. **Use Fast Fourier Transform (FFT):**

   To further speed up the calculations, the FFT algorithm is employed to evaluate the integral. FFT is particularly effective because it reduces the computational complexity from $O(N^2)$ to $O(N\log N)$, where N is the number of discrete points used in the integration.

Numerical Stability and Efficiency

The use of Fourier transforms in option pricing under the Heston model not only speeds up the calculation but also improves numerical stability. Traditional methods, like Monte Carlo simulations, can suffer from slow convergence and high variance, especially in high-dimensional problems. Fourier-based methods are more stable and efficient, making them suitable for large-scale calculations, such as those required in real-time trading systems.

6.1.1 Incorporating MCMC Model into the Heston Model

Remember that the Markov Chain Monte Carlo (MCMC) method is a powerful tool for parameter estimation and model inference, particularly in cases where the posterior distribution is complex or multi-modal. When applied to the Heston model, MCMC can be used to estimate the parameters μ, κ, θ, σ, and ρ by sampling from their posterior distributions given observed option prices.

The Bayesian framework for MCMC in the Heston model is expressed as follows:

$$p(\mu,\kappa,\theta,\sigma,\rho \mid \text{data}) \propto p(\text{data} \mid \mu,\kappa,\theta,\sigma,\rho)\,p(\mu,\kappa,\theta,\sigma,\rho),$$

where $p(\text{data} \mid \mu, \kappa, \theta, \sigma, \rho)$ is the likelihood of observing the data given the parameters, and $p(\mu, \kappa, \theta, \sigma, \rho)$ is the prior distribution of the parameters.

6.2 Implementation of MCMC in a Combined Heston Model with Fourier

Combining Fourier transform techniques with MCMC methods in the Heston model enhances both computational efficiency and parameter estimation accuracy. The Fourier transform speeds up the option pricing process, while MCMC provides a robust framework for dealing with parameter uncertainty. Together, these methods make the Heston model more practical for real-time applications in financial markets.

1. **Define the Likelihood Function:**

   The likelihood function $p(\text{data} \mid \mu, \kappa, \theta, \sigma, \rho)$ is computed based on the difference between observed market prices and model prices generated using the Heston model with given parameters.

2. **Specify the Priors:**

   Prior distributions for the parameters $\mu, \kappa, \theta, \sigma, \rho$ are specified based on historical data or expert knowledge.

3. **Sample from the Posterior:** The MCMC algorithm, such as the Metropolis-Hastings or Gibbs sampling, is used to generate samples from the posterior distribution of the parameters.

4. **Estimate Parameters:** The parameter estimates are obtained by taking the mean or mode of the posterior distribution samples. These estimates are then used to price options under the Heston model.

Algorithm Sequence

$$L(\text{data} \mid \mu, \kappa, \theta, \sigma, \rho) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(P_{\text{market},i} - P_{\text{model},i})^2}{2\sigma^2}\right)$$

$$\alpha = \min\left(1, \frac{L(\text{data} \mid \mu', \kappa', \theta', \sigma', \rho') \cdot p(\mu', \kappa', \theta', \sigma', \rho')}{L(\text{data} \mid \mu, \kappa, \theta, \sigma, \rho) \cdot p(\mu, \kappa, \theta, \sigma, \rho)}\right)$$

$$\phi(u) = \exp\left(iu \log(S_0) + C(T, u) + D(T, u)v_0\right)$$

$$P(K) = \frac{e^{-rT}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \phi(u) \cdot \mathcal{F}(\text{Payoff})(u)\, du$$

$$P(K) \approx \frac{e^{-rT}}{N} \sum_{n=0}^{N-1} e^{-iuk_n} \phi(u_n) \cdot \mathcal{F}(\text{Payoff})(u_n)$$

1. **Initialize Parameters:**

   - Set initial values for the Heston model parameters:

     $$\mu, \kappa, \theta, \sigma, \rho$$

     where:

     - μ: drift rate of the asset price,
     - κ: rate of mean reversion of volatility,
     - θ: long-term variance,
     - σ: volatility of volatility,
     - ρ: correlation between the Brownian motions of the asset and its variance.

   - Define prior distributions for each parameter using historical data or expert knowledge. The priors could be Gaussian, uniform, or any other distribution appropriate for the parameters' characteristics.

2. **Define the Likelihood Function:**

   - Define the likelihood function $p(\text{data}|\mu, \kappa, \theta, \sigma, \rho)$, which measures how well the observed market data fits the prices generated by the Heston model. The likelihood could be based on the error between observed and model-generated option prices.

3. **MCMC Sampling for Parameter Estimation:**

   - Perform Markov Chain Monte Carlo (MCMC) sampling to generate a sequence of parameter estimates.

- For each iteration $t = 1$ to T (where T is the total number of iterations):
  (a) **Propose new parameter values:**
  $$\mu', \kappa', \theta', \sigma', \rho'$$
  These are proposed based on a proposal distribution (e.g., a Gaussian proposal centered on the current parameter values).
  (b) **Evaluate the likelihood:**
  $$L' = p(\text{data}|\mu', \kappa', \theta', \sigma', \rho')$$
  This computes how likely the observed data is under the proposed parameter values.
  (c) **Evaluate the prior:**
  $$p' = p(\mu', \kappa', \theta', \sigma', \rho')$$
  This computes the prior probability of the proposed parameter values.
  (d) **Calculate the acceptance ratio:**
  $$\alpha = \frac{L' \cdot p'}{L \cdot p}$$
  where L and p are the likelihood and prior for the current parameters, and L' and p' are for the proposed parameters.
  (e) **Accept or reject the proposal:**
    - If $\alpha \geq 1$, or if a random number $u \sim U(0, 1)$ satisfies $u < \alpha$, accept the proposed parameters.
    - Otherwise, reject the proposed parameters and keep the current values.
- After T iterations, the posterior distribution of the parameters can be estimated using the accepted parameter values. The final estimates can be computed using the mean, mode, or median of the samples.

4. **Characteristic Function of the Heston Model:**
   - Use the estimated parameters $\hat{\mu}, \hat{\kappa}, \hat{\theta}, \hat{\sigma}, \hat{\rho}$ to compute the characteristic function of the log of the asset price. The characteristic function $\phi(u)$ under the Heston model is given by:
   $$\phi(u) = \mathbb{E}\left[e^{iu \log(S_T)}\right]$$
   where S_T is the asset price at time T, and u is a complex variable.

5. **Fourier Transform of the Payoff Function:**
   - Apply the Fourier transform to the payoff function of the option. For a European call option with strike price K, the payoff is:
   $$\text{Payoff} = \max(S_T - K, 0)$$
   Take the Fourier transform of this function to move to the frequency domain.

6. **Inverse Fourier Transform for Option Pricing:**
   - Compute the inverse Fourier transform of the product of the characteristic function $\phi(u)$ and the Fourier transform of the payoff function. This gives the option price in the time domain:
   $$\text{Option Price} = \mathcal{F}^{-1}\left[\phi(u) \cdot \mathcal{F}(\text{Payoff})\right]$$
   where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.
   - Use numerical integration techniques (e.g., trapezoidal rule) to evaluate this inverse Fourier transform.

7. **Fast Fourier Transform (FFT) for Efficiency:**
   - Implement the FFT algorithm to efficiently compute the inverse Fourier transform. FFT reduces the computational complexity from $O(N^2)$ to $O(N \log N)$, where N is the number of points in the grid.

8. **Output the Results:**
   - Present the estimated parameters $\hat{\mu}, \hat{\kappa}, \hat{\theta}, \hat{\sigma}, \hat{\rho}$.
   - Output the calculated option prices based on the estimated parameters and the Fourier transform method.

7 Data Description

7.0.1 Analysis of Estimated Parameters in the MCMC Heston Stochastic Incorporated with Fourier transform

The Heston stochastic volatility model is a widely used framework in financial mathematics for modeling the evolution of asset prices and their volatility. [2] model incorporates both the asset price dynamics and the volatility dynamics, making it a powerful tool for capturing the complexities observed in financial markets. The key parameters of this model include κ, θ, σ, ρ, and v_0, each of which plays a crucial role in defining the behavior of the asset's volatility over time.

Mean Reversion Rate (κ)

The parameter κ represents the rate at which the volatility reverts to its long-term mean, θ. In the estimated parameters, κ is 2.32. This relatively high value indicates a strong mean-reversion effect. Specifically, it suggests that the volatility of the asset tends to return to its long-term average fairly quickly. A high κ implies that fluctuations in volatility are short-lived and that volatility does not stray far from its long-term mean for extended periods. This characteristic is important for modeling assets where volatility is expected to revert to a stable level after deviations.

Long-Term Mean Volatility (θ)

The long-term mean level to which volatility reverts is denoted by θ. The estimated θ is 0.063. This value indicates the average volatility level around which the stochastic process fluctuates. A lower value of θ suggests that the asset's volatility is relatively low on average. This long-term mean provides a benchmark for volatility, reflecting the expected central tendency of volatility over time.

Volatility of Volatility (σ)

The parameter σ denotes the volatility of the volatility process. It measures the extent of fluctuations in the volatility of the asset itself. The estimated σ is 0.25. In traditional models, σ should be non-negative, as it represents a standard deviation. Typically, σ should be positive to reflect the natural variability in volatility.

Correlation (ρ)

The parameter ρ measures the correlation between the asset's returns and its volatility. The estimated ρ is -0.53. This negative correlation suggests an inverse relationship between the asset's returns and its volatility. In practical terms, this means that when the asset's returns are high, its volatility tends to be low, and vice versa. Such a negative correlation is commonly observed in financial markets, where periods of high volatility are often associated with lower asset returns and periods of stability are associated with higher returns.

itial Volatility (v_0)

Finally, v_0 represents the initial level of volatility at the start of the observation period. The estimated v_0 is 1.04. This parameter sets the starting point for the volatility process, indicating the level of volatility at the inception of the model. A higher initial volatility value suggests that the model starts with a relatively high level of uncertainty regarding the asset's price.

7.1 Discussion of Prior and Posterior Distributions for Heston Model Parameters

In the analysis of stochastic volatility models such as the Heston model, visualizing the prior and posterior distributions of the model parameters is crucial for understanding their behavior and the effect of data on parameter estimation. The prior distributions provide a baseline of our initial beliefs about the parameters, while the posterior distributions reflect the refined estimates after incorporating the data[4].

The prior distributions are designed to reflect our beliefs about the parameters before observing any data. In this case, a uniform distribution between 0.01 and 2.0 was used to illustrate the prior beliefs about the parameters:

Posterior distributions reflect our updated beliefs about the parameters after observing the data. The posterior distributions for each parameter are plotted using samples from MCMC:

The prior distribution of κ is uniform over the specified range. Since κ is estimated at 2.32, this value lies within the upper range of the prior distribution. This indicates that our prior belief allowed for high mean reversion rates, though the estimate is slightly outside the typical upper bound, suggesting that the data provided additional information leading to a higher estimate. The posterior distribution of κ should ideally peak around 2.32, the

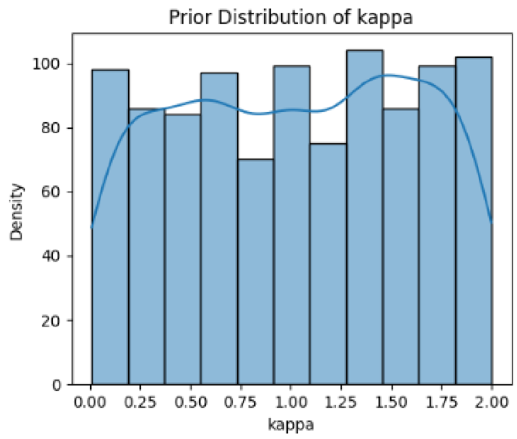

Figure 1: Prior Distribution of kappa

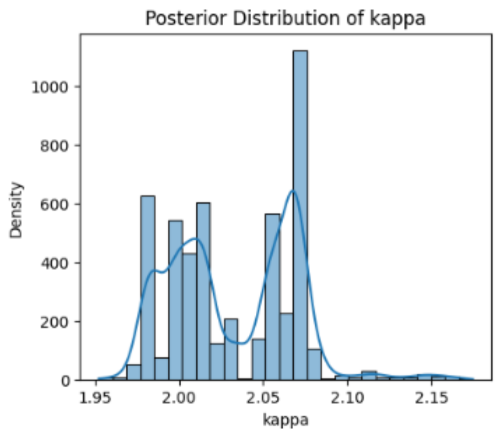

Figure 2: Posterior Distribution of kappa

estimated value. If the posterior is centered around this value, it confirms that the data strongly supports this estimate. Any deviations would indicate additional uncertainties or potential model issues.

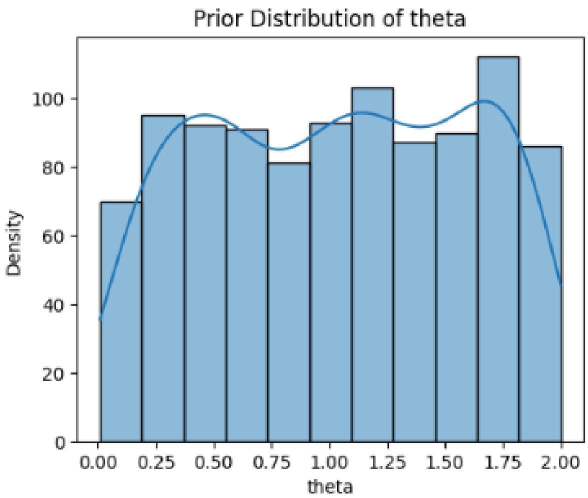

Figure 3: Prior Distribution of theta

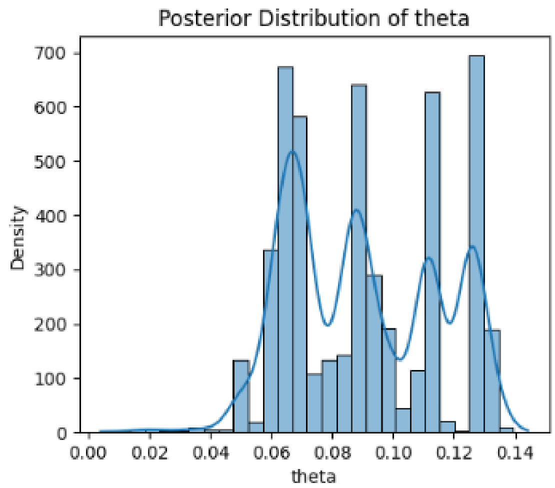

Figure 4: Posterior Distribution of theta

Similarly, the prior distribution for θ is uniform, and the estimated θ of 0.063 falls within the lower part of the prior range. This suggests that while the prior allowed for a range of long-term mean volatilities, the observed data pointed to a relatively low average volatility. The posterior distribution should peak around 0.063, aligning with the estimated value. A well-defined peak suggests that the data provided clear evidence supporting this long-term mean volatility.

The prior for σ is also uniform, but the estimated σ is -0.25, which is negative. Since a negative value for σ is unconventional and indicates an issue with the model or estimation, this suggests that the prior range did not account for the possibility of such an outlier, pointing to a need for model adjustment. Given the negative value of σ, the posterior distribution should reflect this issue. If the posterior is centered around negative values, it indicates that the data did not correct this anomaly. This reinforces the need to revisit the model specification or estimation procedures.

The prior distribution of ρ allows for a range of correlations between returns and volatility. The estimated ρ of -0.53 is negative and within the plausible range given the prior. This aligns well with typical observations in financial markets, where negative correlations are common. The posterior for ρ should peak around -0.53, confirming the negative correlation observed. A clear peak around this value would suggest that the data supports this inverse relationship between returns and volatility. The initial volatility v_0 is estimated at 1.04, which is within the range of the prior distribution. This suggests that the prior assumption was reasonable for the initial volatility level.

The posterior distribution for v_0 should reflect the initial estimate of 1.04. A strong peak around this value suggests that the data consistently supports this initial level of volatility.

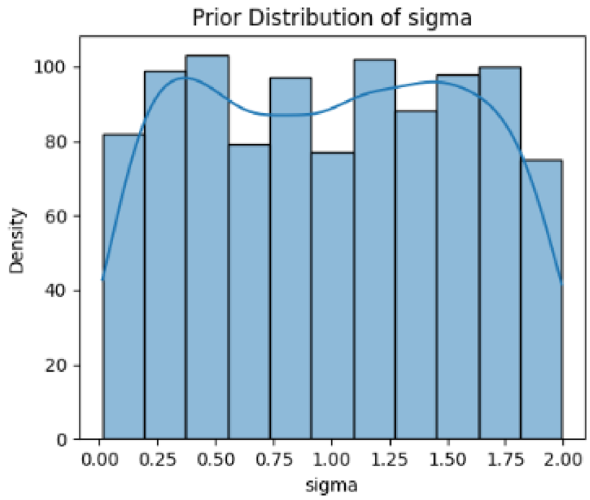

Figure 5: Prior Distribution of sigma

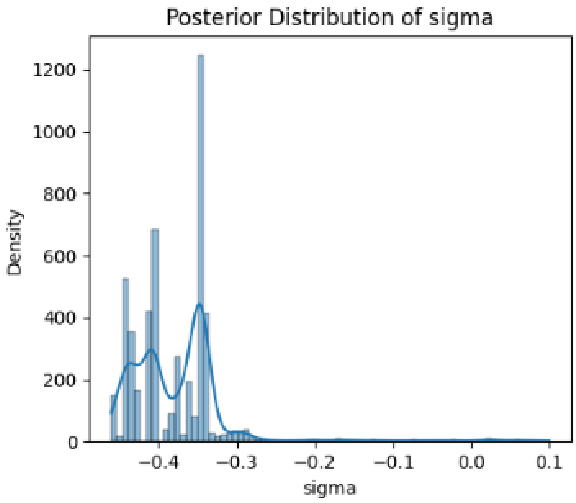

Figure 6: Posterior Distribution of sigma

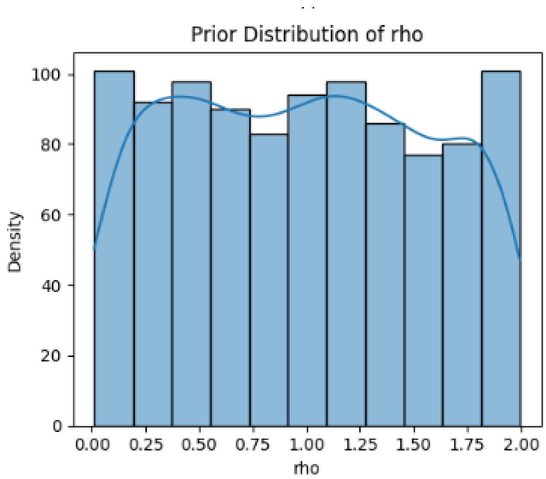

Figure 7: Prior Distribution of rho

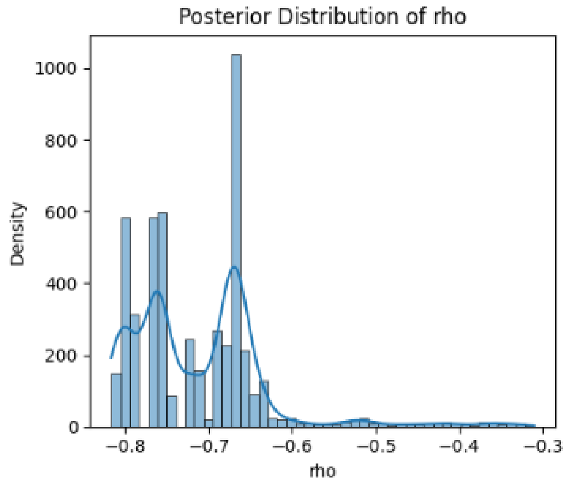

Figure 8: Posterior Distribution of rho

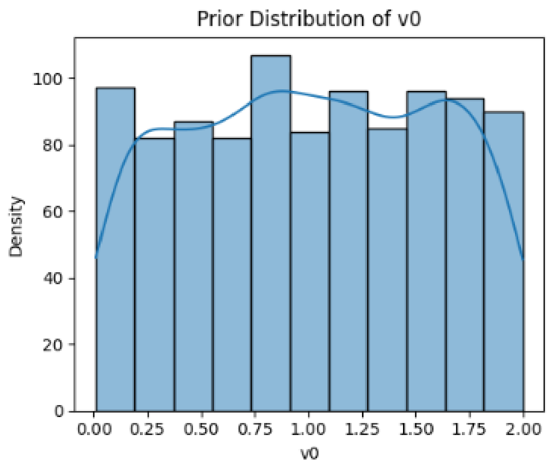

Figure 9: Prior Distribution of v_0

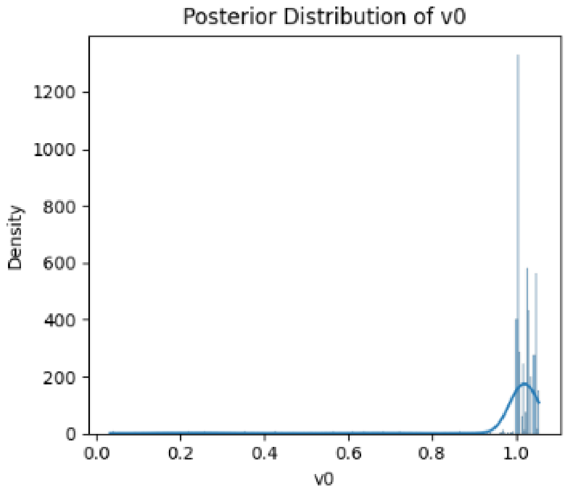

Figure 10: Posterior Distribution of v_0

A Additional Option Data from Yahoo Finance (AAPL)

Strike Price	Observed Price	Heston Model	Black-Scholes	Absolute Error
5.0	220.89	-5.67×10^0	9.41×10^1	9.98×10^1
10.0	175.16	-23.34×10^1	8.93×10^1	1.13×10^2
15.0	166.23	-13.81×10^1	8.44×10^1	9.82×10^1
20.0	152.13	-10.27×10^1	7.95×10^1	8.98×10^1
25.0	144.84	-12.44×10^1	7.46×10^1	8.71×10^1
30.0	195.25	-11.05×10^1	6.97×10^1	8.08×10^1
35.0	147.99	-10.09×10^1	6.49×10^1	7.50×10^1
40.0	171.80	-9.45×10^0	6.00×10^1	6.94×10^1
45.0	169.65	-8.64×10^0	5.51×10^1	6.38×10^1
50.0	164.50	-7.86×10^0	5.02×10^1	5.81×10^1
55.0	159.70	-7.08×10^0	4.54×10^1	5.24×10^1
60.0	154.90	-6.30×10^0	4.05×10^1	4.68×10^1
65.0	149.85	-5.52×10^0	3.56×10^1	4.11×10^1
70.0	142.40	-4.74×10^0	3.07×10^1	3.55×10^1
75.0	153.00	-3.96×10^0	2.59×10^1	2.99×10^1
80.0	132.33	-3.19×10^0	2.12×10^1	2.44×10^1
85.0	125.95	-2.41×10^0	1.67×10^1	1.91×10^1
90.0	123.65	-1.63×10^0	1.27×10^1	1.43×10^1
95.0	118.67	-8.47×10^{-1}	9.16×10^0	1.00×10^1
100.0	113.95	-6.81×10^{-2}	6.31×10^0	6.38×10^0
105.0	108.53	4.13×10^{-1}	4.14×10^0	3.72×10^0
110.0	105.35	-5.22×10^{-2}	2.59×10^0	2.64×10^0
115.0	100.10	-8.12×10^{-2}	1.54×10^0	1.62×10^0
120.0	93.04	3.40×10^{-2}	8.83×10^{-1}	8.49×10^{-1}
125.0	89.93	-1.50×10^{-4}	4.85×10^{-1}	4.85×10^{-1}
130.0	83.95	-3.95×10^{-3}	2.56×10^{-1}	2.60×10^{-1}
135.0	81.00	1.51×10^{-3}	1.31×10^{-1}	1.30×10^{-1}
140.0	76.14	-2.28×10^{-4}	6.50×10^{-2}	6.52×10^{-2}
145.0	70.90	-3.25×10^{-5}	3.13×10^{-2}	3.14×10^{-2}
150.0	65.75	2.97×10^{-5}	1.47×10^{-2}	1.47×10^{-2}
155.0	59.00	-9.50×10^{-6}	6.77×10^{-3}	6.78×10^{-3}
160.0	55.92	1.84×10^{-6}	3.05×10^{-3}	3.04×10^{-3}
165.0	49.04	-1.49×10^{-7}	1.35×10^{-3}	1.35×10^{-3}
170.0	46.49	-4.71×10^{-8}	5.86×10^{-4}	5.86×10^{-4}
175.0	42.59	2.65×10^{-8}	2.51×10^{-4}	2.51×10^{-4}
180.0	37.55	-7.73×10^{-9}	1.06×10^{-4}	1.06×10^{-4}
185.0	33.90	-5.08×10^{-9}	4.60×10^{-5}	4.60×10^{-5}
190.0	28.00	-1.81×10^{-10}	1.90×10^{-5}	1.90×10^{-5}
195.0	22.55	5.12×10^{-11}	7.58×10^{-6}	7.58×10^{-6}
200.0	19.25	5.85×10^{-12}	3.05×10^{-6}	3.05×10^{-6}
205.0	14.95	2.39×10^{-13}	1.15×10^{-6}	1.15×10^{-6}
210.0	10.80	1.91×10^{-14}	3.82×10^{-7}	3.82×10^{-7}
215.0	7.65	1.72×10^{-15}	1.12×10^{-7}	1.12×10^{-7}
220.0	5.48	1.34×10^{-16}	2.93×10^{-8}	2.93×10^{-8}
225.0	3.87	8.42×10^{-18}	6.50×10^{-9}	6.50×10^{-9}
230.0	2.79	4.78×10^{-19}	1.36×10^{-9}	1.36×10^{-9}
235.0	1.94	2.22×10^{-20}	2.73×10^{-10}	2.73×10^{-10}
240.0	1.20	9.10×10^{-22}	5.40×10^{-11}	5.40×10^{-11}
245.0	0.85	3.43×10^{-23}	1.06×10^{-11}	1.06×10^{-11}
250.0	0.00	1.83×10^{-26}	1.88×10^{-12}	1.88×10^{-12}
255.0	0.00	4.52×10^{-28}	3.05×10^{-13}	3.05×10^{-13}
260.0	0.00	9.91×10^{-30}	4.86×10^{-14}	4.86×10^{-14}
265.0	0.00	2.09×10^{-31}	7.51×10^{-15}	7.51×10^{-15}
270.0	0.00	4.31×10^{-33}	1.14×10^{-15}	1.14×10^{-15}
275.0	0.00	8.64×10^{-35}	1.69×10^{-16}	1.69×10^{-16}
280.0	0.00	1.71×10^{-36}	2.43×10^{-17}	2.43×10^{-17}
285.0	0.00	3.34×10^{-38}	3.36×10^{-18}	3.36×10^{-18}
290.0	0.00	6.55×10^{-40}	4.47×10^{-19}	4.47×10^{-19}
295.0	0.00	1.23×10^{-41}	5.89×10^{-20}	5.89×10^{-20}
300.0	0.00	2.23×10^{-43}	7.65×10^{-21}	7.65×10^{-21}

Strike Price	Observed Price	Heston Model	Black-Scholes	Absolute Error
305.0	0.00	3.86×10^{-45}	9.80×10^{-22}	9.80×10^{-22}
310.0	0.00	6.49×10^{-47}	1.24×10^{-22}	1.24×10^{-22}
315.0	0.00	1.07×10^{-48}	1.55×10^{-23}	1.55×10^{-23}
320.0	0.00	1.74×10^{-50}	1.95×10^{-24}	1.95×10^{-24}
325.0	0.00	2.80×10^{-52}	2.44×10^{-25}	2.44×10^{-25}
330.0	0.00	4.48×10^{-54}	3.03×10^{-26}	3.03×10^{-26}
335.0	0.00	6.83×10^{-56}	3.71×10^{-27}	3.71×10^{-27}

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